Journal of Applied Mechanics and Technical Physics, Vol. 36, No. 1, 1995

ASYMPTOTE OF THE BASIC EQUATION FOR PERTURBATION PROPAGATION IN A LOW-VISCOSITY TWO-DIMENSIONAL MEDIUM

D. B. Rokhlin

UDC 532.516

We will assume that the process of perturbation propagation in a viscous medium is described by the equation

$$P\zeta = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \varepsilon^2\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right)\zeta = f(x,y,t).$$

in particular, this is valid for a viscous gas.

At $\varepsilon = 0$ the basic solution for the operator P has a singularity on the front:

$$\zeta = \frac{\theta(t - R)}{2\pi(t^2 - R^2)^{1/2}}$$

(where $R = (x^2 + y^2)^{1/2}$, θ is the Heaviside function). For the case $\varepsilon \neq 0$ it is continuous, and thus must be a function of the boundary layer type in the vicinity of the front as $\varepsilon \to 0$. The present study will construct asymptotic expansions of the fundamental solution for the operator P in terms of the parameter $\omega = t/\varepsilon^2 \to \infty$ in three regions: ahead of the front, behind the front, and in the vicinity of the front.

1. Integral Representation. We will apply to the equation $P\zeta = \delta(x, y, t)$ a Fourier transform over the spatial variables, find the solution of the corresponding ordinary differential equation in analogy to [1, p. 200], and return to the Fourier representation:

$$\zeta = \frac{\theta(t)}{2\pi}\int_0^{+\infty}\exp\left(-\frac{\varepsilon^2}{2}r^2t\right)\frac{\operatorname{sh}\alpha_0(r)t}{\alpha_0(r)}rJ_0(rR)dr$$

($\alpha_0(\rho) = \left(\frac{\varepsilon^4}{4}r^4 - r^2\right)^{1/2}$, J_0 is a Bessel function). Taking the Laplace transform of this expression, we obtain

$$\int_0^{+\infty}\exp(-pt)\zeta dt = \frac{1}{2\pi}\int_0^{+\infty}\frac{rJ_0(rR)dr}{p^2 + (1 + \varepsilon^2p)r^2} = \frac{1}{2\pi(1 + \varepsilon^2p)}K_0\left(\frac{pR}{(1 + \varepsilon^2p)^{1/2}}\right)$$

(where K_0 is a Macdonald function). Here we make use of Fubini's theorem and the expression presented on p. 264 of [2] and Eq. (6.532.4) from [3, p. 692]. Then, using an inversion formula with consideration of the replacement of variables

$$p = (z^2 - 1)/\varepsilon^2 \tag{1.2}$$

and Eq. (9.238.3) of [3, p. 1077] we find

Rostov-on-Don. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, No. 1, pp. 121-129, January-February, 1995. Original article submitted July, 1993; revision submitted February 28, 1994.

$$\zeta = \frac{\omega}{2\pi^{3/2}it}\int_C \frac{\exp(\omega q(z))}{z}\Psi\left(\frac{1}{2},1;\ 2\lambda\omega\left(z-\frac{1}{z}\right)\right)dz, \tag{1.2}$$

where Ψ is a degenerate hypergeometric function; $\lambda = R/t$; $\omega = t/\varepsilon^2$ is a large parameter; the contour C is the image of a vertical line with the replacement of Eq. (1.1); $q(z) = z^2 - 1 - \lambda(z - 1/z)$.

The function $w = z - 1/z$ maps a region $d = \{z{:}\mathrm{Re}z > 0,\ |z| > 1\}$ into the region $d_1' = \{z{:}\mathrm{Re}z > 0\}$, the region $d_2 = \{z{:}\mathrm{Re}z > 0,\ |z| < 1,\ z \notin (0,1)\}$ into the region $d_2' = \{z{:}\mathrm{Re}z < 0,\ z \notin (-\infty, 0)\}$, the boundary of region d_1 onto the imaginary axis, and the interval (0, 1) into the ray $(-\infty, 0)$. Therefore the right semiplane with section [0, 1] is region in which the integrand is holomorphic, and the point $z = 1$ is a logarithmic branching point.

2. Crossing Point and Line of Most Rapid Descent. The crossing points satisfy the equations $q'(z) = 0$:

$$z^3 - \frac{\lambda}{2}z^2 - \frac{\lambda}{2} = 0. \tag{2.1}$$

Equation (2.1) has a single real root $z_1 > 0$ (Descartes's law of signs) and two complex conjugate roots in the left semiplane (Raus–Gruvitz theorem). The latter will not be considered further. We will note the following easily verifiable properties of the root z_1:

a) the crossing point z_1 is simple: $q''(z_1) > 0$;

b) at $\lambda = 1$ $z_1 = 1$;

c) the function $z_1 = z_1(\lambda)$ is monotonically increasing;

d) $q(z_1) < 0$ for $\lambda \neq 1$;

e) if $\mathrm{Im}\, z = 0$, $z > z_1$, then $q'(z) > 0$.

It is clear that z_1 can be calculated by Cardano's formulas. Using a Newton diagram we find

$$z_1 = 1 + (\lambda - 1)/2 + \ldots,\ \lambda \to 1. \tag{2.2}$$

Let $z = \xi + i\eta$. The equation of the line of most rapid descent will be defined from the relationships $\mathrm{Im}q(z) = \mathrm{Im}q(z_1)$, $\mathrm{Re}q(z) < \mathrm{Re}q(z_1)$:

$$(2\xi - \lambda)(\xi^2 + \eta^2) - \lambda = 0. \tag{2.3}$$

it follows from Eq. (2.3) that $\xi > \lambda/2$, the line of most rapid descent is symmetric about the real axis and admits the explicit representation $\xi = \xi(\eta)$. The function $\xi = \xi(\eta)$ is monotonically increasing for $\eta < 0$ and monotonically decreasing for $\eta > 0$. Note also that $\xi(\eta) \to \lambda/2$ as $\eta \to \pm\infty$.

We will denote the line of most rapid descent by L. Let $\rho = |z|$, $\varphi = \arg(z)$. As can easily be shown, for the arcs of circles C_ρ^1, C_ρ^2, located between C and L,

$$\mathrm{Re}q(z) = \rho^2\cos(2\varphi) - \lambda\rho\cos(\varphi) + O(1) < 0$$

for sufficiently large ρ. In accordance with Eq. (13.5.2) of [4, p. 325] the degenerate hypergeometric function will have a power law asymptote for large values of the argument:

$$\Psi\left(\frac{1}{2},1;z\right) = \sum_{k=0}^{N-1}(-1)^k\frac{\left(\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_k}{k!z^{k+1/2}} + R_N(z). \tag{2.4}$$

Here $R_N(z) = O(z^{-N-1/2})$ as $|z| \to \infty$, $|\varphi| < 3\pi/2$, $(1/2)_k = \Gamma(k + 1/2)/\Gamma(1/2)$ is the Pochgammer symbol. Then for $z_1 > 1$ the contour C can be deformed into L. Other deformations of the contour which will be performed below can be similarly justified.

We will now write in general form the basic representation of the function ζ, which we will then concretize. We assume that we have deformed the contour C into some Contour $K = K_\delta \cup (K\backslash K_\delta)$ and that the integral I_δ over $K\backslash K_\delta$ is exponentially small as compared to the integral over K_δ. Expanding the function Ψ with Eq. (2.4) and substituting this expansion in Eq. (1.2), we obtain

$$\zeta = \frac{1}{2\pi^{3/2}it}\sum_{k=0}^{N-1}(-1)^k\frac{\left(\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_k J^{-k-1/2}}{k!(2\lambda)^{k+1/2}\omega^{k-1/2}} + I_N + I_\delta; \tag{2.5}$$

$$J^\nu = \int_{K_\delta}\frac{\exp(\omega q(z))}{z}\left(z - \frac{1}{z}\right)^\nu dz; \tag{2.6}$$

$$I_N = \frac{\omega}{2\pi^{3/2}it}\int_{K_\delta}\frac{\exp(\omega q(z))}{z}R_N\left(2\lambda\omega\left(z - \frac{1}{z}\right)\right)dz. \tag{2.7}$$

We will call J^ν the reference integral.

3. Asymptote Ahead of the Front: $\lambda > 1$, $\omega \to \infty$. In this case we will integrate along the line of most rapid descent. Let $L_\delta \ni z_1$ be a segment of that line of length δ. We take $K = L$, $K_\delta = L_\delta$. Then from the fundamental property of the line of most rapid descent we have $I_\delta = O(\exp(\omega q(z_1) - \omega\gamma))$, $\gamma > 0$.

To calculate reference integral (2.6) we make the replacement

$$q(z) - q(z_1) = -w^2. \tag{3.1}$$

By the theorem of the inverse function, in some vicinity of the origin on the plane, w is defined by the holomorphic function $z = z(w)$, which reduces Eq. (3.1) to the identity:

$$z = z(w) = z_1 + i(2/q''(z_1))^{1/2}w + \ldots$$

The inverse representation has the form

$$w = w(z) = (q(z_1) - q(z))^{1/2} = -i(q''(z_1)/2)^{1/2}(z - z_1) + \ldots \tag{3.2}$$

The image of the contour L_δ for Eq. (3.2) is then the segment of the real axis $[-\alpha, \beta]$, $\alpha, \beta > 0$:

$$J^\nu = \int_{-\alpha}^{\beta}\exp(\omega q(z_1) - \omega w^2)G(w,\nu)dw,\ G(w,\nu) = \frac{(z^2(w) - 1)^\nu}{z^{\nu+1}(w)}\frac{dz}{dw}(w).$$

By Watson's lemma [5, p. 57] we find

$$J^\nu \sim \exp(\omega q(z_1))\sum_{n=0}^{\infty}\frac{\Gamma(n + 1/2)}{\omega^{n+1/2}(2n)!}\frac{\partial^{2n}G}{\partial w^{2n}}(0,\nu). \tag{3.3}$$

Equation (3.3) shows that the terms of series (2.5) form an asymptotic scale. To justify this expansion the following coarse estimate of the residue of Eq. (2.7) is sufficient:

$$|I_N| \leqslant A\exp(\omega q(z_1))\omega^{-N+1/2},$$

where A is independent of ω.

We define the coefficients in the expansion of (3.3) with the Cauchy formula

$$\frac{\partial^n G}{\partial w^n}(0,\nu) = \frac{n!}{2\pi i}\int_{\gamma'}\frac{G(w,\nu)}{w^{n+1}}dw = \frac{n!}{2\pi i}\int_{\gamma''}\frac{(z^2 - 1)^\nu}{w^{n+1}(z)z^{\nu+1}}dz =$$

$$= \lim_{z \to z_1}\frac{d^n}{dz^n}\left(\left(\frac{z - z_1}{w(z)}\right)^{n+1}\frac{(z^2 - 1)^\nu}{z^{\nu+1}}\right),$$

where γ' is a closed contour surrounding the origin in the plane w, while γ'' is its image in the plane z. Having used this formula, we write the main term of expansion (2.5):

$$\zeta = \frac{\exp(\omega q(z_1))}{2\pi(z_1(z_1^2 - 1)q''(z_1)Rt)^{1/2}} + O(\omega^{-1}\exp(\omega q(z_1))). \tag{3.4}$$

it is obvious that the expression thus obtained is non-uniform as $\lambda \to 1 + 0$.

4. Asymptote Behind the Front: $\lambda < 1$, $\omega \to \infty$. Considering the $z_1 < 1$, we take the integration contour S as the lower part of the line of most rapid descent, the segment $[z_1, 1-\delta]$ following the lower boundary of the segment [0, 1], the circle S_δ with center at the point $z = 1$ and radius δ, the segment $[1-\delta, z_1]$ along the upper boundary of the segment, and the upper portion of the line of most rapid descent. Here δ is some small positive number.

We take $K = S$, $K_\delta = S_\delta$ in Eqs. (2.5)-(2.7). It then follows from the properties of the function q(z) that the point $z = 1$ lies on the relief surface Re q(z) above the contour $S \backslash S_\delta$. In as much as $q(1) = 0$, we have $I_\delta = O(\exp(-\omega\gamma))$, $\gamma > 0$. To calculate the reference integral (2.6) we make the replacement $q(z) - q(1) = w$, obtaining

$$J^\nu = \int_{S'_\delta} \exp(\omega w) w^\nu Q(w,\nu)dw, \quad Q(w,\nu) = \frac{(z^2(w) - 1)^\nu}{w^\nu z^{\nu+1}(w)} \frac{dz}{dw}(w).$$

The closed contour S'_δ surrounds the origin in the plane w and moves in the positive direction (counterclockwise). The function Q(w, v) is holomorphic in the vicinity of the point $w = 0$. In analogy to the preceding, with the Cauchy formula we find

$$\frac{\partial^n Q}{\partial w^n}(0,\nu) = \left.\frac{d^n}{d^n z}\right|_{z=1} \left(\frac{z^n}{(z - \lambda)^{\nu+n+1}(z + 1)^{n+1}}\right) \tag{4.1}$$

$$= \sum_{r+j+s=n} (-1)^{r+j} \frac{(n + j)!n!}{r!j!s!(n - s)!} \frac{1}{2^{n+j+1}(1 - \lambda)^{\nu+n+r+1}} \frac{\Gamma(\nu + n + r + 1)}{\Gamma(\nu + n + 1)}$$

(where r, j, s are non-negative integers).

We will now expand Q(w, v) in a Taylor series in the vicinity of the origin and make use of Watson's lemma for integrals over a loop [5, p. 272]:

$$J^\nu = 2\pi i \sum_{n=0}^{M-1} \frac{\omega^{-n-\nu-1}}{n!\Gamma(-\nu - n)} \frac{\partial^n Q}{\partial w^n}(0,\nu) + I^1_M. \tag{4.2}$$

Here $I^1_M = O(\omega^{-M-\nu-1})$. It follows from Eq. (4.2) that all terms of series (2.5) are of the order of $\omega^0 = 1$. After substitution of Eq. (4.2) in Eq. (2.5) and regrouping, we have

$$\zeta = \frac{1}{\pi^{1/2} t} \sum_{n=0}^{M-1} \sum_{k=0}^{N-1} \alpha^k_n \omega^{-n} + I_N + I^2_{MN} + I_\delta, \tag{4.3}$$

where $\alpha^k_n = (-1)^k \dfrac{\left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_k \frac{\partial^n Q}{\partial w^n}(0,-k-1/2)}{k!n!(2\lambda)^{k+1/2}\Gamma(k+1/2-n)}$; $I^2_{MN} = O(\omega^{-M})$.

We will assume that $\lim_{N\to\infty} I_N = 0$, $\lim_{N\to\infty} I^2_{MN} = O(\omega^{-M})$. Then, transforming to the limit as $N \to \infty$ in Eq. (4.3), we obtain the asymptotic expansion

$$\zeta \sim \sum_{n=0}^{\infty} \zeta_n \omega^{-n}; \tag{4.4}$$

$$\zeta_n = \frac{1}{\pi^{1/2} t} \sum_{k=0}^{\infty} \alpha_n^k . \tag{4.5}$$

For 1/3 < λ < 1 the series of Eq. (4.5) converge and are expressible in terms of a hypergeometric function:

$$\zeta_n = \frac{\left(\frac{1-\lambda}{2\lambda}\right)^{1/2}}{\pi^{1/2} t 2^{n+1} (1-\lambda)^{n+1}} \sum_{r+j+s=n} (-1)^j \frac{(n+j)!}{r! j! s! (n-s)!} \times$$
$$\times \frac{1}{2^j (1-\lambda)^r \Gamma(1/2-n-r)^2} F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}-n-r;\frac{\lambda-1}{2\lambda}\right), \tag{4.6}$$
$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k, \ |z| < 1.$$

Setting n = 0 in Eq. (4.6) and using Eq (5.2.11.16) from [6, p. 710], we find

$$\zeta_0 = \frac{1}{2\pi t} \frac{1}{(2\lambda)^{1/2} (1-\lambda)^{1/2}} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{k!} \left(\frac{\lambda-1}{2\lambda}\right)^k = \frac{1}{2\pi (t^2 - R^2)^{1/2}}. \tag{4.7}$$

Thus, ζ_0 is the fundamental solution of the degenerate problem.

Thus, if our assumption regarding the behavior of the residues I_N, I^2_{MN}, is true, the asymptotic expressions of Eqs. (4.4), (4.5) are valid for 1/3 < λ < 1.

Independently of the above, we will now obtain analogous expressions for the case 0 < λ < 1/3. To do this, we transform the integrand of Eq. (1.2). Let $f_0(z)$ be an integer function. In accordance with Eq. (13.1.6) of [4, p. 322] we have

$$\Psi\left(\frac{1}{2},1;z\right) = -\pi^{-1/2} {}_1F_1\left(\frac{1}{2};1;z\right) \ln(z) + f_0(z); \tag{4.8}$$
$${}_1F_1(a;b;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}. \tag{4.9}$$

We will integrate over the contour S. Since the integral of a holomorphic function over the closed contour S_δ is equal to zero, after substitution of Eq. (4.8) into Eq. (1.2) and integration of series (4.9) we obtain

$$\zeta = -\frac{\omega}{2\pi^2 i t} \sum_{k=0}^{N-1} \frac{\left(\frac{1}{2}\right)_k}{(k!)^2} (2\lambda\omega)^k \frac{\partial J^\nu}{\partial \nu}\Big|_{\nu=k} + I^3_N + I_\delta. \tag{4.10}$$

Here I^3_N is the integral of the series residue. Note that $\lim_{N\to\infty} I^3_N = 0$. Differentiating asymptote (4.2) with respect to ν, we find

$$\frac{\partial J^\nu}{\partial \nu}\Big|_{\nu=k} = 2\pi t \sum_{n=0}^{M-1} (-1)^{n+k+1} \frac{(n+k)!}{n! \omega^{n+k+1}} \frac{\partial^n Q}{\partial w^n}(0,k) + I^4_M, \tag{4.11}$$

where $I^4_M = O(\omega^{-M-k-1})$ and the following relationships are used:

$$\frac{1}{\Gamma(-k-n)} = 0, \ \frac{\partial}{\partial \nu}\Big|_{\nu=k} \frac{1}{\Gamma(-\nu-n)} = (-1)^{n+k+1}(n+k)!.$$

It follows from Eq. (4.11) that all the terms of series (4.10) are of the order of $\omega^0 = 1$. We substitute Eq. (4.11) in Eq. (4.10):

$$\zeta = \frac{1}{\pi t}\sum_{n=0}^{M-1}\sum_{k=0}^{N-1}\beta_n^k\omega^{-n} + I_N^3 + I_{M,N}^5 + I_\delta. \tag{4.12}$$

Here

$$\beta_n^k = (-1)^{n+k}\frac{(n+k)!\left(\frac{1}{2}\right)_k}{n!(k!)^2}(2\lambda)^k\frac{\partial^n Q}{\partial^n w}(0,k).$$

We assume that $\lim_{N\to\infty} I_{M,N}^5 = O(\omega^{-M})$. After transition in Eq. (4.12) to the limit as $N \to \infty$ we arrive at expansion (4.4), where

$$\zeta_n = \frac{1}{\pi t}\sum_{k=0}^{\infty}\beta_n^k. \tag{4.13}$$

In analogy to the preceding, with the aid of Eq. (4.1) we find that for $0 < \lambda < 1/3$ the series for ζ_0 converges to the fundamental solution of the degenerate problem, while the remaining series of Eq. (4.13) can be expressed in terms of the hypergeometric function:

$$\begin{aligned}\zeta_n = {}& \frac{(-1)^n}{\pi t 2^{n+1}(1-\lambda)^{n+1}}\sum_{r+j+s=n}(-1)^{r+j}\frac{(n+j)!}{r!j!s!(n-s)!}\\ &\times\frac{\Gamma(n+r+1)}{2^j(1-\lambda)^r}\,{}_2F_1\left(\frac{1}{2},n+r+1;1;\frac{2\lambda}{\lambda-1}\right).\end{aligned} \tag{4.14}$$

Taking into account Eq. (15.3.7) of [4, p. 373],

$$\begin{aligned}{}_2F_1\left(\frac{1}{2},n+r+1;1;\frac{2\lambda}{\lambda-1}\right) = {}& \frac{\Gamma(n+r+1/2)}{\pi^{1/2}\Gamma(n+r+1)}\left(\frac{1-\lambda}{2\lambda}\right)^{1/2}\\ &\times{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{1}{2}-n-r;\frac{\lambda-1}{2\lambda}\right),\end{aligned}$$

we conclude that the functions of Eq. (4.14) are analytical continuations along λ of the functions of Eq. (4.6) on the interval (0, 1/3). This fact, together with the correct expression for ζ_0 confirms the validity of our assumption as to the behavior of the residues I_N, $I_{M,N}^2$, $I_{M,N}^5$ to a certain degree.

5. Uniform Expansion in the Vicinity of the Front. $\lambda \approx 1$, $\omega \to \infty$. In Eqs. (2.5)-(2.7) we take $K_\delta = C_\delta$, $K = C_\delta \cup (L \backslash L_\delta)$. Here C_δ is a small arc, intersecting the real axis to the right of the valley point z_1 and the branching point $z = 1$, while the contour L_δ is as defined in Section 3.

We will calculate the reference integral by Bleistein's method [7]. It will be convenient to integrate over the entire contour K in Eq. (2.6) and assign the corresponding exponential error to I_δ.

We will consider the equation

$$q(z) = -(w^2/2 + bw), \tag{5.1}$$

where $b = i(2\,|\,q(z_1)\,|\,)^{1/2}\text{sign}(\lambda - 1)$. Within a certain vicinity of the most rapidly descending curve, containing the contour C_δ and the point $z = 1$, a holomorphic function is defined which transforms Eq. (5.1) to the identity:

$$w = w(z) = -b + (b^2 - 2q(z))^{1/2}. \tag{5.2}$$

It will suffice to determine the sign of the root in Eq. (5.2) on the most rapidly descending curve:

$$\text{sign}((b^2 - 2q(z))^{1/2}) = \text{sign}(\text{Re}z),\ z \in L.$$

Upon transformation of Eq. (5.2) the point z = 1 transforms to the origin of the coordinate system, the point $z = z_1$ into the point w = −b, and the contour K into the contour K_1, coincident with the straight line Im w = −b outside some vicinity of the origin. The reference integral then transforms to:

$$J^{\nu} = \int_{K_1} \exp\left(-\omega\left(\frac{w^2}{2} + bw\right)\right) w^{\nu} g(w,\nu)\,dw. \tag{5.3}$$

Here $g(w, \nu) = \dfrac{(z^2(w) - 1)^{\nu}}{z(w)^{\nu+1}} \dfrac{1}{w^{\nu}} \dfrac{dz}{dw}$ is a holomorphic function.

To calculate the integral of Eq. (5.3) we represent the function g(w, ν) in the form

$$g(w,\nu) = \gamma_0(\nu) + \gamma_1(\nu)w + w(w + b)g_1(w,\nu).$$

Then

$$J^{\nu} = \omega^{-(\nu+1)/2}\gamma_0(\nu)U_{\nu}(b\omega^{1/2}) + \omega^{-(\nu+2)/2}\gamma_1(\nu)U_{\nu+1}(b\omega^{1/2}) + J_1^{\nu};$$
$$U_{\nu}(s) = \int_{K_1} \exp\left(-\left(\frac{w^2}{2} + sw\right)\right) w^{\nu}dw = (2\pi)^{1/2}\exp\left(-i\nu\frac{\pi}{2}\right)\exp\left(\frac{s^2}{4}\right)D_{\nu}(-is), \tag{5.4}$$

where D_{ν} is a parabolic cylinder function. Here we make use of Eq. (3.462.3) from [3, p. 352] with consideration of the fact that for Re $\nu > -1$ the contour K_1 can be deformed to the real axis. For Re $\nu \leq -1$, Eq. (5.4) is valid by the principle of analytical extension.

Integrating by parts, we find

$$J_1^{\nu} = \int_{K_1} \exp\left(-\omega\left(\frac{w^2}{2} + bw\right)\right) w^{\nu+1}(w + b)g_1(w,\nu)dw$$
$$= \omega^{-1}\int_{K_1} \exp\left(-\omega\left(\frac{w^2}{2} + bw\right)\right) w^{\nu}\left((\nu + 1)g_1(w,\nu) + w\frac{dg_1}{dw}(w,\nu)\right)dw.$$

Using the representation $(\nu + 1)g_1 + w\dfrac{dg_1}{dw} = \gamma_2 + \gamma_3 w + (w + b)wg_2$ and continuing the iteration process, we arrive at the expansion [7]

$$J^{\nu} = \frac{U_{\nu}(b\omega^{1/2})}{\omega^{(\nu+1)/2}}\left[\sum_{m=0}^{M-1} \frac{\gamma_{2m}}{\omega^m} + O(\omega^{-M})\right] + \frac{U_{\nu+1}(b\omega^{1/2})}{\omega^{(\nu+2)/2}}\left[\sum_{m=0}^{M-1} \frac{\gamma_{2m+1}}{\omega^m} + O(\omega^{-M})\right]; \tag{5.5}$$

$$\gamma_0(\nu) = g(0,\nu),\ \gamma_1(\nu) = (g(0,\nu) - g(-b,\nu))/b,$$
$$\gamma_{2m}(\nu) = (\nu + 1)g_m(0,\nu), \tag{5.6}$$
$$\gamma_{2m+1}(\nu) = (\nu + 1)(g_m(0,\nu) - g_m(-b,\nu))/b + \frac{dg_m}{dw}(-b,\nu).$$

The functions $g_{m+1}(w)$, $m \geq 1$ are defined by the recursion formula

$$(\nu + 1)g_m(w,\nu) + w\frac{dg_m}{dw}(w,\nu) = \gamma_{2m}(\nu) + \gamma_{2m+1}(\nu)w + w(w + b)g_{m+1}(w,\nu).$$

In order to determine the order of magnitude of the quantities in Eq. (2.5), we find asymptotic expansions of the function $U_{\nu}(b\omega^{1/2})$, $\omega \to \infty$. In accordance with Eqs. (9.246.1), (9.246.2) from [3, p. 1079], we have

$$U_{\nu}(b\omega^{1/2}) \sim (2\pi)^{1/2}\exp\left(-i\nu\frac{\pi}{2}\right)\exp(-\omega|b|^2/2)|b|^{\nu}\omega^{\nu/2},\ \lambda > 1; \tag{5.7}$$

$$U_\nu(b\omega^{1/2}) \sim \frac{2\pi}{\Gamma(-\nu)}\exp\left(-i\nu\frac{\pi}{2}\right)|b|^{-\nu-1}\omega^{-(\nu+1)/2},\ \lambda < 1. \tag{5.8}$$

In as much as $b = O(\lambda - 1)$ (see Eq. (2.2)) and the coefficients γ_k are finite functions for small b, we conclude that the terms of the series of Eq. (2.5) form an asymptotic scale on the interval $\lambda \in [1 - \omega^{-\mu}, 1 + \lambda_0]$, where $\lambda_0 = \text{const}$, μ being an arbitrary positive number. In fact, from Eqs. (5.5), (5.7), (5.8) we find

a) $\omega^{-k+1/2}J^{-k-1/2} \sim B_1(b)\exp(-\omega|b|^2/2)\omega^{-k},\ \lambda - 1 \in (0, \lambda_0]$;

b) $\omega^{-k+1/2}J^{-k-1/2} \sim B_2|b|^{k-1/2},\ \lambda - 1 \in [-\omega^{-\mu_1}, -\omega^{-\mu_2}]$;

c) $\omega^{-k+1/2}J^{-k-1/2} \sim B_3\omega^{-k/2+1/4},\ \lambda - 1 \in [-\omega^{-\mu_3}, 0]$,

where

$$\mu_1 > 0,\ \mu_1 < \mu_2 < 1/2,\ \mu_3 \geq 1/2,$$

$$B_1(b) = (2\pi)^{1/2}\exp\left(i\left(k + \frac{1}{2}\right)\frac{\pi}{2}\right)|b|^{-k-1/2}\left(\gamma_0\left(-k - \frac{1}{2}\right) - b\gamma_1\left(-k - \frac{1}{2}\right)\right),$$

$$B_2 = 2\pi/\Gamma\left(k + \frac{1}{2}\right)\exp\left(i\left(k + \frac{1}{2}\right)\frac{\pi}{2}\right)\gamma_0\left(-k - \frac{1}{2}\right),$$

$$B_3 = \gamma_0\left(-k - \frac{1}{2}\right)U_{-k-1/2}(-ic_0),\ c_0 = \lim_{\omega\to\infty}|b|\omega^{1/2}.$$

Restricting ourselves to the main terms of the expansions in Eqs. (2.5), (5.5), we have

$$\zeta \sim \frac{1}{(2\pi)^{3/2}\lambda^{1/2}it}\left(\gamma_0\left(-\frac{1}{2}\right)U_{-1/2}(b\omega^{1/2})\omega^{1/4} + \gamma_1\left(-\frac{1}{2}\right)U_{1/2}(b\omega^{1/2})\omega^{-1/4}\right). \tag{5.9}$$

We will calculate the coefficients $\gamma_0(-1/2)$, $\gamma_1(-1/2)$ with Eq. (5.6):

$$\gamma_0\left(-\frac{1}{2}\right) = \left(\frac{-b}{2q'(1)}\right)^{1/2},$$
$$\gamma_1\left(-\frac{1}{2}\right) = \frac{1}{b}\left(\left(\frac{-b}{2q'(1)}\right)^{1/2} - \frac{i}{z_1^{1/2}(z_1^2 - 1)^{1/2}}\left(\frac{-b}{q''(z)_1}\right)^{1/2}\right). \tag{5.10}$$

Equation (5.9) defines the main term of the uniform expansion ζ on the interval $\lambda \in [1 - \omega^{-\mu}, 1 + \lambda_0]$, $\mu > 0$. Equation (3.4) can be derived therefrom, but not the expressions of Section 4, since in the latter $\lambda < 1$ is fixed.

Note that on the front itself the solution grows as $\varepsilon \to 0$. In fact, expanding the indefiniteness in Eq. (5.10) and using Eq. (19.3.5) of [4, p. 496], we obtain

$$\zeta \sim \frac{\omega^{1/4}}{(2\pi)^{3/2}\,it)}\gamma_0\left(-\frac{1}{2}\right)U_{-1/2}(0) = \frac{\omega^{1/4}}{2^{9/4}\pi^{1/2}\Gamma\left(\frac{3}{4}\right)t},\ \lambda = 1,\ \omega \to \infty.$$

REFERENCES

1. V. S. Vladimirov, Equations of Mathematical Physics [in Russian], Nauka, Moscow (1988).
2. G. Dech, Handbook of Practical Applications of the Laplace and Z-Transforms [in Russian], Nauka, Moscow (1971).
3. I. S. Gradshtein and I. M. Ryzhik, Tables of Integrals, Sums, and Products [in Russian], Nauka, Moscow (1971).
4. M. Abramovits and Y. Stigan (eds.), Handbook of Special Functions with Formulas, Graphs, and Mathematical Tables [in Russian], Nauka, Moscow (1979).

5. M. V. Fedoryuk, Asymptotes: Integrals and Series [in Russian], Nauka, Moscow (1987).
6. A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, Integrals and Series [in Russian], Nauka, Moscow (1981).
7. N. Bleistein, "Uniform asymptotic expansions of integrals with stationary point near algebraic singularity," Comm. Pure Appl. Math., **19**, No. 4 (1966).